Quantum groupoids associated to universal dynamical R-matrices

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Abstract

By using twist construction, we obtain a quantum groupoid $\mathcal{D}\otimes_q U_q\mathfrak{g}$ for any simple Lie algebra $\mathfrak{g}$. The underlying Hopf algebroid structure encodes all the information of the corresponding elliptic quantum group-the quasi-Hopf algebras as obtained by Fronsdal, Arnaudon et. al. and Jimbo et. al..

Groupoïdes quantiques associés à des R-matrices dynamiques universelles

Résumé En utlisant une construction twistée, nous obtenons, pour toute algèbre de Lie simple $\mathfrak{g}$, un groupoïde quantique $\mathcal{D}\otimes_q U_q\mathfrak{g}$. La structure d'algébroïde de Hopf sous-jacente encode toute l'information contenue dans le groupe quantique elliptique correspondant—cf. les quasi-algèbres de Hopf obtenues par Fronsdal, Arnaudon et. al. et Jimbo et. al..

Version française abrégée

Il y a, depuis peu, un intérêt grandissant pour l'équation de Yang-Baxter quantique universelle, aussi connue sous le nom d'équation de Geravais-Neveu-Felder :

$$R_{12}(\lambda + \hbar h^{(3)})R_{13}(\lambda)R_{23}(\lambda + \hbar h^{(1)}) = R_{23}(\lambda)R_{13}(\lambda + \hbar h^{(2)})R_{12}(\lambda). \tag{1}$$

Ici, $R(\lambda)$ est une fonction méromorphe de η^* vers $U_q\mathfrak{g}\otimes U_q\mathfrak{g}$, où $U_q\mathfrak{g}$ est un groupe quantique quasi-triangulaire, et $\eta \subset \mathfrak{g}$ est une sous-algèbre de Cartan. Cette équation apparaît naturellement dans de nombreux contextes de la physique mathématique tels que le théorie de Liouville quantique, l'équation de Knizhnik-Zamolodchikov-Bernard quantique ou encore le modèle de Caloger-Moser quantique. Une approche de cette équation est celle Babelon et. al. [3] qui utilise la théorie des quasi-algèbres de Hopf. Considèrons une fonction méromorphe $F : \eta^* \longrightarrow U_q\mathfrak{g}\otimes U_q\mathfrak{g}$ telle que $F(\lambda)$ est inversible pour tout λ. Posons $R(\lambda) = F_{21}(\lambda)^{-1}RF_{12}(\lambda)$, où $R \in U_q\mathfrak{g}\otimes U_q\mathfrak{g}$ est la R-matrice universelle standard pour le groupe quantique $U_q\mathfrak{g}$. On peut vérifier que $R(\lambda)$ satisfait l'équation (1) quand $F(\lambda)$ est de poids nul, et satisfait la condition de cocycle décalé suivante :

$$(\Delta_0\otimes id)F(\lambda)F_{12}(\lambda + \hbar h^{(3)}) = (id\otimes\Delta_0)F(\lambda)F_{23}(\lambda), \tag{2}$$

où Δ_0 est le co-produit dans $U_q\mathfrak{g}$. Si nous supposons de plus que

$$(\epsilon_0 \otimes id)F(\lambda) = 1; \quad (id \otimes \epsilon_0)F(\lambda) = 1, \tag{3}$$

où ϵ_0 est l'application co-unité, on peut former un groupe quantique elliptique qui est une famille de quasi-algèbres de Hopf $(U_q\mathfrak{g}, \Delta_\lambda)$ paramétrisée par $\lambda \in \eta^*$: $\Delta_\lambda = F(\lambda)^{-1}\Delta F(\lambda)$.

Le but de cet article est de montrer que l'équation (2) dérive aussi naturellement de l'équation "twistorielle" d'un groupoïde quantique. Ceci conduit à interpréter un groupe quantique elliptique comme un groupoïde quantique. Tout d'abord nous introduisons une construction générale twistée pour les algébroïdes de Hopf. De manière analogue au cas des algèbres de Hopf, nous montrons le

Théorème A *Soient $(H, R, \alpha, \beta, m, \Delta, \epsilon)$, un algébroïde de Hopf d'ancre τ, et $\mathcal{F} \in H \otimes_R H$ un twisteur. Alors, $(H, R_\mathcal{F}, \alpha_\mathcal{F}, \beta_\mathcal{F}, m, \Delta_\mathcal{F}, \epsilon)$ est un algébroïde de Hopf admettant τ pour ancre. La catégorie monoïdale de H-modules correspondante est équivalente à celle correspondant à $(H, R, \alpha, \beta, m, \Delta, \epsilon)$.*

Maintenant, considérons $H = \mathcal{D} \otimes U_q\mathfrak{g}[[\hbar]]$ $(q = e^\hbar)$, où $\mathcal{D}$ est l'algèbre des opérateurs différentiels sur η^*. H est de manière naturelle un algébroïde de Hopf. Nous lui appliquons la construction twistée :

Théorème B *Soit $F(\lambda) \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ une fonction méromorphe de poids nul. Alors, $\mathcal{F} = F(\lambda)\Theta \in H \otimes_R H$ est un twisteur (c.à.d. satisfait l'équation (7) et (8)) ssi les équations (2) et (3) tiennent. Ici, Θ est défini par l'équation (10).*

Nous notons $\mathcal{D} \otimes_q U_q\mathfrak{g}$, le groupoïde quantique obtenu en twistant $\mathcal{D} \otimes U_q\mathfrak{g}[[\hbar]]$ via $\mathcal{F}$. Comme conséquence immédiate du théorème A, nous avons le

Théorème C *En tant que catégorie monoïdale, la catégorie des $\mathcal{D} \otimes_q U_q\mathfrak{g}$-modules est équivalente à celle des $\mathcal{D} \otimes U_q\mathfrak{g}[[\hbar]]$-modules, et est donc une catégorie monoïdale tressée.*

Nous observons que $\mathcal{D} \otimes_q U_q\mathfrak{g}$ est co-associatif en tant qu'algébroïde de Hopf, bien que $(U_q\mathfrak{g}, \Delta_\lambda)$ ne le soit pas. Dès lors, la construction de $\mathcal{D} \otimes_q U_q\mathfrak{g}$ consiste, d'une certaine manière, à établir la co-associativité en élargissant l'algèbre $U_q\mathfrak{g}$ par produit tensoriel avec la partie dynamique $\mathcal{D}$. Le lien entre ce groupoïde quantique et les quasi-algèbres de Hopf est, dans un certain sens, analogue à celui qu'il y a entre un fibré vectoriel et ses fibres.

1   Introduction

Recently, there is an increasing interest in the so called quantum universal dynamical Yang-Baxter equation, also known as Geravais-Neveu-Felder equation:

$$R_{12}(\lambda + \hbar h^{(3)})R_{13}(\lambda)R_{23}(\lambda + \hbar h^{(1)}) = R_{23}(\lambda)R_{13}(\lambda + \hbar h^{(2)})R_{12}(\lambda). \tag{4}$$

Here $R(\lambda)$ is a meromorphic function from η^* to $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ $(q = e^\hbar)$, $\mathfrak{g}$ is a complex semi-simple Lie algebra with Cartan subalgebra η, and $U_q\mathfrak{g}$ is a quasi-triangular quantum group. This equation arises naturally from various contexts in mathematical physics, including quantum Liouville theory, quantum Knizhnik-Zamolodchikov-Bernard equation, and quantum Caloger-Moser model. One approach to this equation due to Babelon et. al. [3] is to use Drinfeld's theory of quais-Hopf algebras. Consider a meromorphic function $F : \eta^* \longrightarrow U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ such that $F(\lambda)$ is invertible for all λ. Set $R(\lambda) = F_{21}(\lambda)^{-1}RF_{12}(\lambda)$, where

$R \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is the standard universal R-matrix for the quantum group $U_q\mathfrak{g}$. One can check that $R(\lambda)$ satisfies Equation (4) if $F(\lambda)$ is of zero weight, and satisfies the following shifted cocycle condition:

$$(\Delta_0 \otimes id)F(\lambda)F_{12}(\lambda + \hbar h^{(3)}) = (id \otimes \Delta_0)F(\lambda)F_{23}(\lambda), \tag{5}$$

where Δ_0 is the coproduct of $U_q\mathfrak{g}$. If moreover we assume that

$$(\epsilon_0 \otimes id)F(\lambda) = 1; \quad (id \otimes \epsilon_0)F(\lambda) = 1, \tag{6}$$

where ϵ_0 is the counit map, one can form an elliptic quantum group, which is a family of quasi-Hopf algebras $(U_q\mathfrak{g}, \Delta_\lambda)$ parameterized by $\lambda \in \eta^*$: $\Delta_\lambda = F(\lambda)^{-1}\Delta F(\lambda)$. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, a solution to Equations (5) and (6) was obtained by Babelon [2] in 1991. For general simple Lie algebras, solutions were recently found independently by Arnaudon et. al. [1] and Jimbo et. al. [7] based on the approach of Fronsdal [6].

The purpose of this Note is to show that Equation (5) also arises naturally from the "twistor" equation of a quantum groupoid. This leads to another interpretation of an elliptic quantum group, namely as a quantum groupoid. The notion of quantum groupoids, as a quantization of Lie bialgebroids, was introduced in [8] as a general framework unifying quantum groups and star-products on Poisson manifolds. Roughly speaking, our construction goes as follows. Instead of $U_q\mathfrak{g}$, we start with the algebra $H = \mathcal{D} \otimes U_q\mathfrak{g}[[\hbar]]$ ($q = e^\hbar$), where $\mathcal{D}$ is the algebra of holomorphic differential operators on η^*. H is no longer a Hopf algebra. Instead it is a Hopf algebroid as a tensor product of Hopf algebroids $\mathcal{D}[[\hbar]]$ and $U_q\mathfrak{g}$. The shifted cocycle condition for $F(\lambda)$ is then shown to be equivalent to the twistor equation for this Hopf algebroid. Using this twistor, we obtain a quantum groupoid $\mathcal{D} \otimes_q U_q\mathfrak{g}$. We note that $\mathcal{D} \otimes_q U_q\mathfrak{g}$ is co-associative as a Hopf algebroid, although Δ_λ is not co-associative. The construction of $\mathcal{D} \otimes_q U_q\mathfrak{g}$ is in some sense to restore the co-associativity by enlarging the algebra $U_q\mathfrak{g}$ by tensoring the dynamical part $\mathcal{D}$. The relation between this quantum groupoid and the quasi-Hopf algebras $(U_q\mathfrak{g}, \Delta_\lambda)$ is, in a certain sense, analogous to that between a fiber bundle and its fibers. We expect that this quantum groupoid will be useful in understanding elliptic quantum groups, especially their representation theory. The physical meaning of this quantum groupoid, however, still needs to be explored.

The first part of the Note is devoted to the introduction of twist construction for Hopf algebroids. Then we retain to this particular example of universal dynamical R-matrices in the second part. We note that a different Hopf algebroid approach to the quantum dynamical Yang-Baxter equation was also obtained by Etingof and Varchenko [4].

2   Twist construction

Let $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ be a Hopf algebroid (see [8] for detail). By $End_k R$, we denote the algebra of linear endomorphisms of R (over the ground ring k). It is clear that $End_k R$ is an (R, R)-bimodule, where R acts on it from both left and right by multiplications. An *anchor* map is an (R, R)-bimodule map $\tau : H \longrightarrow End_k R$ satisfying certain axioms, which, roughly speaking, means that τ is a Hopf algebroid morphism. More precisely, for $x \in H$ and $a \in R$, we denote by $x(a)$ the element in R defined by $x(a) = \tau(x)(a)$, and by φ_α and φ_β the maps $(H \otimes_R H) \otimes R \longrightarrow H$ defined, respectively, by $\varphi_\alpha(x \otimes_R y \otimes a) = x(a) \cdot y$, and $\varphi_\beta(x \otimes_R y \otimes a) = x \cdot y(a)$. Here $x, y \in H$, $a \in R$, and the dot $\cdot$ denotes the (R, R)-action on H. Note that φ_α and φ_β are well defined since τ is an (R, R)-bimodule map.

Then we require that (i). $\tau : H \longrightarrow End_k R$ is an algebra homomorphism preserving the identities; (ii). $x(1_R) = \epsilon x, \quad \forall x \in H$; (iii). $\varphi_\alpha(\Delta x \otimes a) = x\alpha(a)$ and $\varphi_\beta(\Delta x \otimes a) = x\beta(a), \quad \forall x \in H, a \in R$.

For $U_q\mathfrak{g}$, since $R = k$ and $End_k R \cong k$, one can simply take τ as the counit. On the other hand, if H is the Hopf algebroid $\mathcal{D}$ of differential operators over a manifold M (see Example 2.1 in [8]), the usual action of differential operators on functions is an anchor map. More generally, if $H = \mathcal{D} \otimes U_q\mathfrak{g}$ considered as a Hopf algebroid over $R = C^\infty(M)$, then the map: $\tau(D \otimes u)(f) = (\epsilon_0 u)D(f), \quad \forall D \in \mathcal{D}, u \in U_q\mathfrak{g}, f \in C^\infty(M)$ is an anchor. Here ϵ_0 is the counit of the Hopf algebra $U_q\mathfrak{g}$.

Given a Hopf algebroid $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ with anchor τ, let $\mathcal{F}$ be an element in $H \otimes_R H$ satisfying:

$$(\Delta \otimes_R id_H)\mathcal{F} \cdot \mathcal{F}^{12} = (id_H \otimes_R \Delta)\mathcal{F} \cdot \mathcal{F}^{23} \quad \text{in} \quad H \otimes_R H \otimes_R H; \quad \text{and} \tag{7}$$

$$(\epsilon \otimes_R id_H)\mathcal{F} = 1_H; \quad (id_H \otimes_R \epsilon)\mathcal{F} = 1_H, \tag{8}$$

where $\mathcal{F}^{12} = \mathcal{F} \otimes 1 \in (H \otimes_R H) \otimes H$, $\mathcal{F}^{23} = 1 \otimes \mathcal{F} \in H \otimes (H \otimes_R H)$, and in Equation (8) we have used the identification: $R \otimes_R H \cong H \otimes_R R \cong H$. Define $\alpha_\mathcal{F}, \beta_\mathcal{F} : R \longrightarrow H$, respectively, by $\alpha_\mathcal{F}(a) = \varphi_\alpha(\mathcal{F} \otimes a)$, $\beta_\mathcal{F}(a) = \varphi_\beta(\mathcal{F} \otimes a), \forall a \in R$. For any $a, b \in R$, set $a *_\mathcal{F} b = \tau(\alpha_\mathcal{F}(a))(b)$. More explicitly, if $\mathcal{F} = \sum_i x_i \otimes_R y_i$ for $x_i, y_i \in H$, then $\alpha_\mathcal{F}(a) = \sum_i x_i(a) \cdot y_i$, $\beta_\mathcal{F}(a) = \sum_i x_i \cdot y_i(a)$, and $a *_\mathcal{F} b = \sum_i x_i(a)y_i(b), \forall a, b \in R$. Using Equations (7) and (8), one can prove the following:

Proposition 2.1 (i). $(R, *_\mathcal{F})$ is an associative algebra, and $1_R *_\mathcal{F} a = a *_\mathcal{F} 1_R = a, \forall a \in R$.

(ii). $\alpha_\mathcal{F} : R_\mathcal{F} \longrightarrow H$ is an algebra homomorphism, and $\beta_\mathcal{F} : R_\mathcal{F} \longrightarrow H$ is anti-homomorphism. Here $R_\mathcal{F}$ stands for the algebra $(R, *_\mathcal{F})$.

Moreover, it is not difficult to check that Equation (7) also implies that $\mathcal{F}(\beta_\mathcal{F}(a) \otimes 1 - 1 \otimes \alpha_\mathcal{F}(a)) = 0$ in $H \otimes_R H, \quad \forall a \in R$. As an immediate consequence, we have

Lemma 2.2 Let M_1 and M_2 be any H-modules. Then $\mathcal{F}^\#(m_1 \otimes_{R_\mathcal{F}} m_2) = \mathcal{F} \cdot (m_1 \otimes m_2)$, $m_1 \in M_1$, and $m_2 \in M_2$, is a well defined linear map. Here and in the sequel, by H-modules we always mean left H-modules.

We say that $\mathcal{F}$ is *invertible* if $\mathcal{F}^\#$ is a vector space isomorphism for any H-modules M_1 and M_2. In this case, in particular we can take $M_1 = M_2 = H$ so that we obtain an isomorphism

$$\mathcal{F}^\# : H \otimes_{R_\mathcal{F}} H \longrightarrow H \otimes_R H. \tag{9}$$

Definition 2.3 An element $\mathcal{F} \in H \otimes_R H$ is called a twistor if it is invertible and satisfies both Equations (7) and (8).

Now assume that $\mathcal{F}$ is a twistor. Define a new coproduct $\Delta_\mathcal{F} : H \longrightarrow H \otimes_{R_\mathcal{F}} H$ by $\Delta_\mathcal{F} = \mathcal{F}^{-1}\Delta\mathcal{F}$, which means that $\Delta_\mathcal{F}(x) = \mathcal{F}^{\#^{-1}}(\Delta(x)\mathcal{F}), \forall x \in H$.

Theorem A Assume that $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ is a Hopf algebroid with anchor τ, and $\mathcal{F} \in H \otimes_R H$ a twistor. Then $(H, R_\mathcal{F}, \alpha_\mathcal{F}, \beta_\mathcal{F}, m, \Delta_\mathcal{F}, \epsilon)$ is a Hopf algebroid, which still admits τ as an anchor. Its

4

corresponding monoidal category of H-modules is equivalent to that of $(H, R, \alpha, \beta, m, \Delta, \epsilon)$.

We say that $(H, R_{\mathcal{F}}, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, m, \Delta_{\mathcal{F}}, \epsilon)$ is obtained from $(H, R, \alpha, \beta, m, \Delta, \epsilon)$ by twisting via $\mathcal{F}$.

Example 2.1 Let $H = \mathcal{D}(M)[[\hbar]]$ be equipped with the standard Hopf algebroid structure over the base $R = C^{\infty}(M)[[\hbar]]$. As indicated earlier in this section, H admits a natural anchor map. A twistor $\mathcal{F}$ of the form: $\mathcal{F} = 1 \otimes_R 1 + \hbar B_1 + \cdots \in H \otimes_R H[[\hbar]]$ is equivalent to a $*$-product on M. The corresponding twisted Hopf algebroid was described explicitly in [8].

Remark If $\mathcal{F}_1 \in H \otimes_R H$ is a twistor to the Hopf algebroid H, and $\mathcal{F}_2 \in H \otimes_{R_{\mathcal{F}_1}} H$ a twistor for the twisted Hopf algebroid $H_{\mathcal{F}_1}$, then the Hopf algebroid obtained by twisting H via $\mathcal{F}_1$ then via $\mathcal{F}_2$ is equivalent to that obtained by twisting via $\mathcal{F}_1 \mathcal{F}_2$. Here $\mathcal{F}_1 \mathcal{F}_2 \in H \otimes_R H$ is understood as $\mathcal{F}_1^{\#}(\mathcal{F}_2)$, where $\mathcal{F}_1^{\#} : H \otimes_{R_{\mathcal{F}_1}} H \longrightarrow H \otimes_R H$ is the map as defined in Equation (9).

3 Universal dynamical R-matrices

As in the introduction, $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{C}$ with Cartan subalgebra η, and $U_q\mathfrak{g}$ is a quasi-triangular quantum group. In this section, $C(\eta^*)$ always denotes the space of meromorphic functions on η^*, and $\mathcal{D}$ the algebra of holomorphic differential operators on η^*. Let us consider $H = \mathcal{D} \otimes U_q\mathfrak{g}[[\hbar]]$, where for simplicity we assume that $q = e^{\hbar}$. Then H becomes a Hopf algebroid over the base algebra $R = C(\eta^*)[[\hbar]]$ in a standard way. Its coproduct and counit are denoted respectively by Δ and ϵ. We fix a basis of η, say $\{h_1, \cdots, h_k\}$. Let $\{\xi_1, \cdots, \xi_k\}$ be its dual basis. This defines a coordinate system $(\lambda_1, \cdots, \lambda_k)$ on η^*.

Set

$$\theta = \sum_{i=1}^{k} (\frac{\partial}{\partial \lambda_i} \otimes h_i) \in H \otimes H, \quad \text{and} \quad \Theta = \exp \hbar \theta \in H \otimes H. \tag{10}$$

Note that θ, and hence Θ, is independent of the choice of the basis of η. The following fact can be easily verified.

Lemma 3.1 Θ *satisfies Equations (7) and (8).*

In other words, Θ is a twistor for the Hopf algebroid H. As we shall see below, it is this Θ that links a shifted cocycle $F(\lambda)$ and a Hopf algebroid twistor of H.

Theorem B *Assume that $F \in C(\eta^*, U_q\mathfrak{g} \otimes U_q\mathfrak{g})$ is of zero weight: $[F(\lambda), 1 \otimes h + h \otimes 1] = 0$, $\forall \lambda \in \eta^*, h \in \eta$. Then $\mathcal{F} = F(\lambda)\Theta \in H \otimes_R H$ is a twistor (i.e. satisfies Equations (7) and (8)) iff Equations (5) and (6) hold.*

Now assume that $F(\lambda)$ is a solution to Equations (5) and (6) so that we can form a quantum groupoid by twisting $\mathcal{D} \otimes U_q\mathfrak{g}[[\hbar]]$ via $\mathcal{F}$. The resulting quantum groupoid is denoted by $\mathcal{D} \otimes_q U_q\mathfrak{g}$ ($q = e^{\hbar}$).

As an immediate consequence of Theorem A, we have

Theorem C *As a monoidal category, the category of $\mathcal{D}\otimes_q U_q\mathfrak{g}$-modules is equivalent to that of $\mathcal{D}\otimes U_q\mathfrak{g}[[\hbar]]$-modules, and therefore is a braided monoidal category.*

One can describe $\mathcal{D}\otimes_q U_q\mathfrak{g}$ more explicitly.

Proposition 3.2 (i). $f *_{\mathcal{F}} g = fg$, $\forall f, g \in C(\eta^*)$. *I.e., $R_{\mathcal{F}}$ is isomorphic to $C(\eta^*)[[\hbar]]$ with pointwise multiplication;*

(ii). $\alpha_{\mathcal{F}} f = \exp\left(\hbar \sum_{i=1}^k h_i \frac{\partial}{\partial \lambda_i}\right) f = \sum_{1 \le i_1, \cdots, i_n \le k} \frac{\hbar^n}{n!} \frac{\partial^n f}{\partial \lambda_{i_1} \cdots \lambda_{i_n}} h_{i_1} \cdots h_{i_n}$, $\forall f \in C(\eta^*)$;

(iii). $\beta_{\mathcal{F}} f = f$, $\forall f \in C(\eta^*)$.

To describe the relation between $\Delta_{\mathcal{F}}$ and the quasi-Hopf algebra coproducts Δ_λ. We need a "projection" map from $H \otimes_{R_{\mathcal{F}}} H$ to $C(\eta^*, U_q\mathfrak{g} \otimes U_q\mathfrak{g})$. This can be defined as follows. Let $Ad_\Theta : H \otimes H \longrightarrow H \otimes H$ be the adjoint operator: $Ad_\Theta w = \Theta w \Theta^{-1}$, $\forall w \in H \otimes H$. There exists an obvious projection from $H \otimes H$ to $C(\eta^*, U_q\mathfrak{g} \otimes U_q\mathfrak{g})$, which is just taking the 0th-order component. Now composing Ad_Θ with this projection, one obtains a map from $H \otimes H$ to $C(\eta^*, U_q\mathfrak{g} \otimes U_q\mathfrak{g})$. It is not difficult to see that this map descends to a map $H \otimes_{R_{\mathcal{F}}} H \longrightarrow C(\eta^*, U_q\mathfrak{g} \otimes U_q\mathfrak{g})$, which is denoted by T.

Proposition 3.3 *Let $i : U_q\mathfrak{g} \longrightarrow H$ be the natural inclusion. Then $\Delta_\lambda = T \circ \Delta_{\mathcal{F}} \circ i$.*

Remark We may replace θ in Equation (10) by $\tilde{\theta} = \frac{1}{2} \sum_{i=1}^k \left(\frac{\partial}{\partial \lambda_i} \otimes h_i - h_i \otimes \frac{\partial}{\partial \lambda_i}\right) \in H \otimes H$ and set $\tilde{\Theta} = \exp \hbar \tilde{\theta} \in H \otimes H$. One can show that $\tilde{\mathcal{F}} = F(\lambda)\tilde{\Theta}$ satisfies Equation (7) is equivalent to the following twisted cocycle condition for $F(\lambda)$:

$$(\Delta_0 \otimes id)F(\lambda)F_{12}(\lambda + \frac{1}{2}\hbar h^{(3)}) = (id \otimes \Delta_0)F(\lambda)F_{23}(\lambda - \frac{1}{2}\hbar h^{(1)}).$$

Then $R(\lambda) = F_{21}(\lambda)^{-1} R F_{12}(\lambda)$ satisfies

$$R_{12}(\lambda + \frac{1}{2}\hbar h^{(3)})R_{13}(\lambda - \frac{1}{2}\hbar h^{(2)})R_{23}(\lambda + \frac{1}{2}\hbar h^{(1)}) = R_{23}(\lambda - \frac{1}{2}\hbar h^{(1)})R_{13}(\lambda + \frac{1}{2}\hbar h^{(2)})R_{12}(\lambda - \frac{1}{2}\hbar h^{(3)}). \quad (11)$$

In fact, both Θ and $\tilde{\Theta}$ are obtained from the deformation quantization of the canonical symplectic structure $T^*\eta^*$ using the normal order and the Weyl order respectively, so they are equivalent. This indicates that solutions to Equation (4) and Equation (11) should be equivalent as well.

Finally, when $F(\lambda)$ is the shifted cocycle obtained in [1] [7], the classical limit (see [8]) of $\mathcal{D}\otimes_q U_q\mathfrak{g}$ is the coboundary Lie bialgebroid (A, Λ), where $A = T\mathfrak{h}^* \times \mathfrak{g}$, $\Lambda = \sum_{i=1}^k \left(\frac{\partial}{\partial \lambda_i} \wedge h_i\right) + r(\lambda) \in \Gamma(\wedge^2 A)$, and $r(\lambda)$ is the standard dynamical r-matrix on $\mathfrak{g}$: $r(\lambda) = \frac{1}{2} \sum_{\alpha \in \Delta} \coth(\frac{1}{2} \ll \alpha, \lambda \gg) E_\alpha \wedge E_{-\alpha}$. Here $\ll, \gg$ is the Killing form of $\mathfrak{g}$, Δ is the set of roots of $\mathfrak{g}$ with respect to η, the E_α and $E_{-\alpha}$'s are root vectors, and $\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ is the hyperbolic cotangent function. As an consequence, we conclude that this coboundary Lie bialgebroid is quantizable in the sense of [8].

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